ELEN E3401: Electromagnetics

Spring 2025

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Lecture #19

Source free Maxwell Equations

Maxwell's Equations: (with $\tilde{\rho}_v = 0, \vec{J}=0$)

$$\vec{\nabla} \cdot \tilde{E} = 0 \qquad \vec{\nabla} \cdot \tilde{H} = 0$$

$$\vec{\nabla} \times \tilde{E} = -j\omega\mu\tilde{H} \qquad \vec{\nabla} \times \tilde{H} = j\omega\epsilon_c\tilde{E}$$

$\boxed{\nabla^2\tilde{E} + \omega^2\mu\epsilon_c\tilde{E} = 0}$ Homogeneous wave equation $\gamma^2 = -\omega^2\mu\epsilon_c$

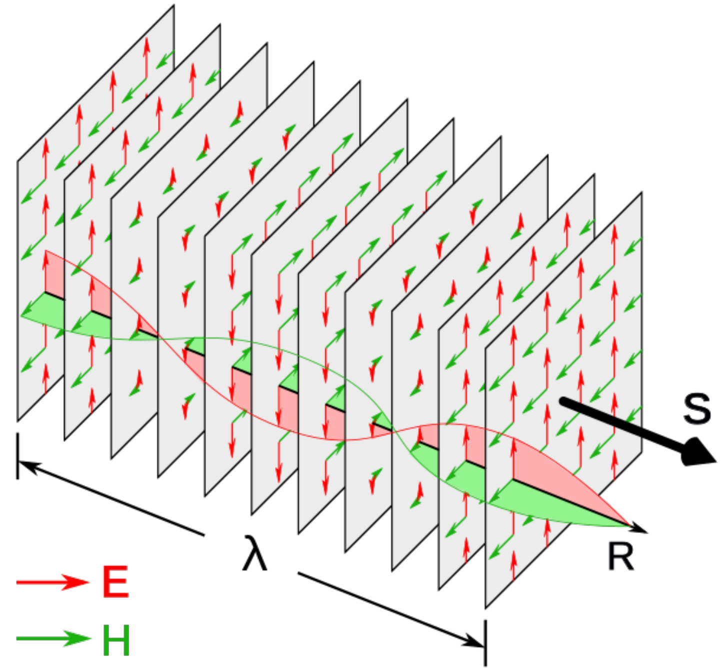
$\boxed{\nabla^2\tilde{E} + k^2\tilde{E} = 0}$ Wave equation → lossless $k = \omega\sqrt{\mu\epsilon}$

Uniform plane wave - lossless

Consider non-conducting, $\sigma = 0$, therefore lossless → $\epsilon_c = \epsilon$

Phase velocity, u_p Wavelength $= \lambda$ Angular frequency $= \omega$

$$\boxed{\nabla^2 \tilde{E} + k^2 \tilde{E} = 0}$$



Consider electric field phasor:

$$\tilde{E} = \hat{x}\tilde{E}_x + \hat{y}\tilde{E}_y + \hat{z}\tilde{E}_z$$

Each vector component must separately = 0

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E}_x = 0$$

Similarly for $\tilde{E}_y$ and $\tilde{E}_z$

Uniform plane wave - lossless

$\vec{E}$ and $\vec{H}$ have uniform properties at all points on infinite plane

If plane wave on *x*-*y* plane → means $\vec{E}$ and $\vec{H}$ do not vary with *x* or *y*

$$\frac{\partial \tilde{E}_x}{\partial x} = 0 \qquad \frac{\partial \tilde{E}_y}{\partial y} = 0$$

Component wave equation simplifies:

$$\frac{d^2 \tilde{E}_x}{dz^2} + k^2 \tilde{E}_x = 0$$

→ Plane wave has no $\vec{E}$ or $\vec{H}$ field components along direction of propagation

For phasor $\tilde{E}_x$ → general wave equation solution:

$$\tilde{E}_x(z) = \tilde{E}_x^+(z) + \tilde{E}_x^-(z) = E_{x0}^+ e^{-jkz} + E_{x0}^- e^{jkz}$$

Constants obtained by
boundary conditions

Why $\tilde{E}_z = 0$?

Consider: $\vec{\nabla} \times \tilde{H} = j\omega\epsilon\tilde{E}$

$\hat{z}$ component: $\hat{z}\left(\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\right) = \hat{z} j\omega\epsilon\tilde{E}_z$

$\frac{\partial \tilde{H}_y}{\partial x} = 0$ $\frac{\partial \tilde{H}_x}{\partial y} = 0$

Therefore $\tilde{E}_z = 0$

We can also take $\vec{\nabla} \times \left(\vec{\nabla} \times \tilde{H}\right)$ and obtain wave equation for $\tilde{H}$

$$\boxed{\nabla^2\tilde{H} - \gamma^2\tilde{H} = 0}$$

Uniform plane wave propagation

$E_{x0}^{+}e^{-jkz}$ Wave amplitude E_{x0}^{+} traveling in +z direction

$E_{x0}^{-}e^{jkz}$ Wave amplitude E_{x0}^{-} traveling in -z direction

Assume $\tilde{E}_y = 0$ and only +z direction ($E_{x0}^{-} = 0$)

$$\tilde{E}(z) = \hat{x}\tilde{E}_x^{+}(z) = \hat{x}E_{x0}^{+}e^{-jkz}$$

Uniform plane wave propagation

$$\tilde{E}(z) = \hat{x}\tilde{E}_x^+(z) = \hat{x}E_{x0}^+ e^{-jkz}$$

Find magnetic field, $\tilde{H}$ associated with this plane wave:

$$\vec{\nabla} \times \tilde{E} = -j\omega\mu\tilde{H}$$

$$\vec{\nabla} \times \tilde{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \tilde{E}_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu(\hat{x}\tilde{H}_x + \hat{y}\tilde{H}_y + \hat{z}\tilde{H}_z)$$

$$\tilde{E}_y = \tilde{E}_z = 0$$

Uniform plane wave propagation

$$\vec{\nabla} \times \tilde{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \tilde{E}_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu(\hat{x}\widetilde{H}_x + \hat{y}\widetilde{H}_y + \hat{z}\widetilde{H}_z)$$

Uniform plane wave, traveling in + $\hat{z}$ direction:

$$\frac{\partial E_x^+(z)}{\partial x} = \frac{\partial E_x^+(z)}{\partial y} = 0 \qquad \text{Uniform plane wave}$$

$$\hat{x}(0) - \hat{y}\underbrace{\left(-\frac{\partial \tilde{E}_x^+(z)}{\partial z}\right)}_{\text{Only non-zero term}} + \hat{z}\underbrace{\left(-\frac{\partial \tilde{E}_x^+(z)}{\partial y}\right)}_{0}$$

$$\widetilde{H}_x = 0 \qquad \widetilde{H}_y = \frac{1}{-j\omega\mu}\frac{\partial \tilde{E}_x^+(z)}{\partial z} \qquad \widetilde{H}_z = \frac{1}{-j\omega\mu}\frac{\partial \tilde{E}_x^+(z)}{\partial y} = 0$$

(only non-zero term)

Uniform plane wave propagation

$$\tilde{E}(z) = \hat{x}\tilde{E}_x^+(z) = \hat{x}E_{x0}^+ e^{-jkz}$$

$$\vec{\nabla} \times \tilde{E} = -\hat{y}\left(-\frac{\partial E_x^+(z)}{\partial z}\right) = -\hat{y}\left(-(-jk)E_{x0}^+ e^{-jkz}\right) = \hat{y}\left(-jkE_{x0}^+ e^{-jkz}\right)$$

$$\widetilde{H}_y(z) = \frac{1}{-j\omega\mu}\left(-jkE_{x0}^+ e^{-jkz}\right) = \underbrace{\frac{k}{\omega\mu}E_{x0}^+}e^{-jkz} = H_{y0}^+ e^{-jkz}$$

$$H_{y0}^+ = \frac{k}{\omega\mu}E_{x0}^+$$

$$\widetilde{H}_y(z) = H_{y0}^+ e^{-jkz}$$

Uniform plane wave propagation

Recall how V_0^+ and I_0^+ related by characteristic impedance, Z_0:

$$
\begin{cases}
\tilde{V}(z) = V_0{}^+ e^{-j\beta z} + V_0{}^- e^{j\beta z} \\[2ex]
\tilde{I}(z) = \dfrac{V_0{}^+}{Z_0} e^{-j\beta z} - \dfrac{V_0{}^-}{Z_0} e j^{\beta z}
\end{cases}
$$

$\vec{E}$ and $\vec{H}$ related by <u>intrinsic impedance</u>:

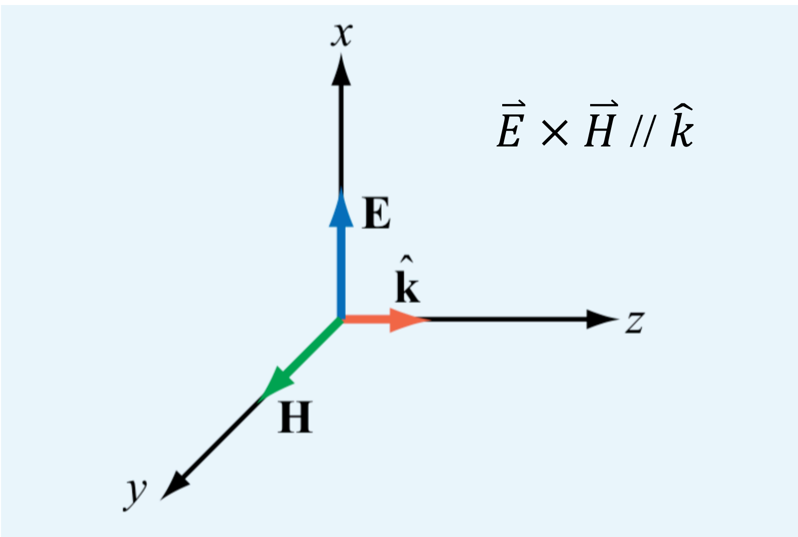
$$\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \ [\Omega] \qquad k = \omega\sqrt{\mu\epsilon}$$

+z propagating plane wave with $\vec{E}$ along $\hat{x}$:

$$\tilde{E}(z) = \hat{x}\tilde{E}_x^+(z) = \hat{x}E_{x0}^+ e^{-jkz} \qquad \tilde{H}_x(z) = \frac{\hat{y}\tilde{E}_x^+(z)}{\eta} = \frac{\hat{y}E_{x0}^+}{\eta} e^{-jkz}$$

$$
\boxed{
\begin{aligned}
\tilde{\mathbf{E}}(z) &= \hat{\mathbf{x}}\tilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-jkz}, \\
\tilde{\mathbf{H}}(z) &= \hat{\mathbf{y}}\frac{\tilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta} e^{-jkz}.
\end{aligned}
}
$$

Transverse Electromagnetic (TEM) Wave



General E_{x0}^{+} is complex, with mag and phase:

$$E_{x0}^{+} = |E_{x0}^{+}| e^{j\varphi^{+}} \qquad \text{(phase angle, } \varphi\text{)}$$

Instantaneous electric field:

$$\vec{E}(z,t) = Re\left[\tilde{E}(z) e^{j\omega t}\right]$$

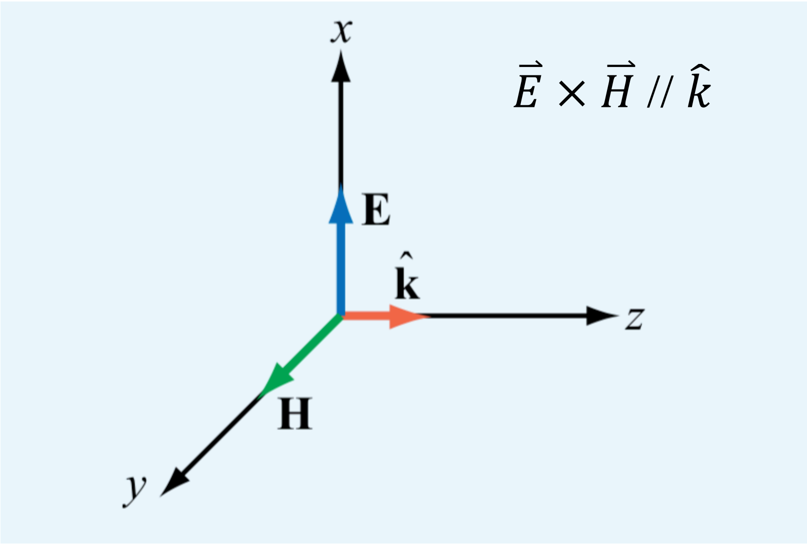
$$\vec{E}(z,t) = \hat{x} |E_{x0}^{+}| \cos(\omega t - kz + \varphi^{+}) \text{ [V/m]}$$

Instantaneous magnetic field:

$$\vec{H}(z,t) = Re\left[\tilde{H}(z) e^{j\omega t}\right]$$

$$\vec{H}(z,t) = \hat{y} \frac{|E_{x0}^{+}|}{\eta} \cos(\omega t - kz + \varphi^{+}) \text{ [A/m]}$$

Transverse Electromagnetic (TEM) Wave



<u>$\tilde{E}$ and $\tilde{H}$ → in phase</u>

Phase velocity: $u_p = \dfrac{\omega}{k} = \dfrac{\omega}{\omega\sqrt{\mu\epsilon}} = \dfrac{1}{\sqrt{\mu\epsilon}}$ $\qquad \lambda = \dfrac{2\pi}{k} = \dfrac{u_p}{f}$

In vacuum: $\epsilon = \epsilon_0, \mu = \mu_0$ $\qquad u_p = c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}} = 3 \times 10^8 \dfrac{m}{s}$

Intrinsic impedance of free space: $\eta = \eta_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 377 \sim 120\pi \;\; [\Omega]$

Example: EM plane wave in air

Electric field of 1 MHz plane wave propagating in +z direction, points along *x*-axis

At t=0 and z=50m → field reaches a peak value: 1.2 π rad/m

Obtain and plot $\vec{E}(z,t)$ and $\vec{H}(z,t)$ at t=0.

Example: EM plane wave in air

For f = 1MHz: $\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300m$

Wave number: $k = \frac{2\pi}{\lambda} = \frac{2\pi}{300}\ \mathrm{rad}/m$

$$\vec{E}(z,t) = \hat{x}|E_{x0}^{+}|\cos(\omega t - kz + \varphi^{+})$$

$$\tilde{E}(z) = \hat{x}E_{x0}^{+}e^{-jkz} \rightarrow \vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right]$$

$$\vec{E}(z,t) = \hat{x}(1.2\ \pi)\cos(\underbrace{2\pi 10^6}_{\omega = 2\pi f}t - \frac{2\pi}{300}z + \varphi^{+})$$

Max E → when cos() = 1 or argument = 0

At t = 0, z = 50m the cosine argument: $2\pi 10^6(0) - \frac{2\pi}{300}(50) + \varphi^{+} = 0 \rightarrow \quad \varphi^{+} = \frac{\pi}{3}$

$$\vec{E}(z,t) = \hat{x}1.2\ \pi\cos\left(2\pi 10^6 t - \frac{2\pi}{300}z + \frac{\pi}{3}\right)\ [mV/m]$$

Example: EM plane wave in air

$$\vec{E}(z,t) = \hat{x}1.2\,\pi\cos\left(2\pi 10^6 t - \frac{2\pi}{300}z + \frac{\pi}{3}\right)\ [mV/m]$$

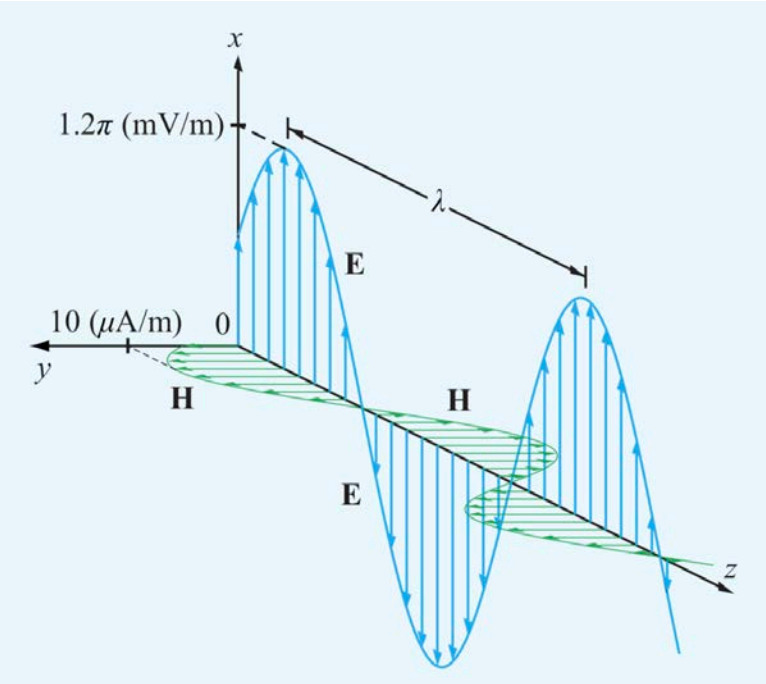
Now we obtain $\vec{H}(z,t)$: $\vec{H}(z,t) = \dfrac{\hat{y}E(z,t)}{\eta_0}$ $\eta_0 \sim 120\pi$ $\dfrac{1.2\pi}{120\pi} = 0.01$

$$\vec{H}(z,t) = \hat{y}0.010\cos\left(2\pi 10^6 t - \frac{2\pi}{300}z + \frac{\pi}{3}\right)\ [mA/m]$$

At $t = 0$:

$$\vec{E}(z,0) = \hat{x}1.2\,\pi\cos\left(\frac{2\pi}{300}z - \frac{\pi}{3}\right)\ [mV/m]$$

$$\vec{H}(z,0) = \hat{y}0.01\cos\left(\frac{2\pi}{300}z - \frac{\pi}{3}\right)\ [mA/m]$$



General relation between $\vec{E}$ and $\vec{H}$

Uniform plane wave traveling in $\hat{k}$ direction:

RHR: $\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E}$

$\tilde{E} = -\eta\hat{k} \times \tilde{H}$

$\left.\begin{array}{l} \\ \end{array}\right\}$ Apply to lossy medium

For Any TEM Wave

Wave propagating in +z with $\vec{E}$ along $\hat{x}$:

Direction of propagation: $\hat{k} = \hat{z}$

$$\tilde{E} = \hat{x}\tilde{E}_x^+(z)$$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \frac{1}{\eta}(\hat{z} \times \hat{x})\tilde{E}_x^+(z) = \frac{\hat{y}\tilde{E}_x^+(z)}{\eta}$$

Same result we obtained

General relation between $\vec{E}$ and $\vec{H}$

Our simple example of *x*-polarized $\vec{E}$ traveling in -$\hat{z}$ direction ($\hat{k} = -\hat{z}$):

$$\tilde{E} = \hat{x}\tilde{E}_x^-(z) = \hat{x}E_{x0}^- e^{jkz}$$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \frac{1}{\eta}(-\hat{z} \times \hat{x})\tilde{E}_x^-(z) = \frac{-\hat{y}\tilde{E}_x^-(z)}{\eta}$$

$$\tilde{H} = -\hat{y}\frac{E_{x0}^-}{\eta}e^{jkz}$$

Now $\tilde{H}$ points along $-\hat{y}$ direction

General relation between $\vec{E}$ and $\vec{H}$

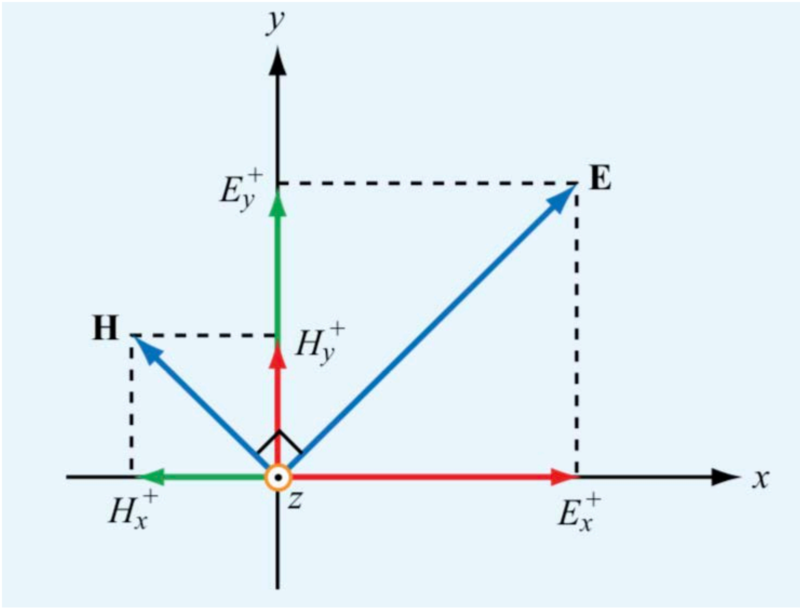
Generalize plane wave propagating along $+z$ will have $\hat{x}$ and $\hat{y}$ components:

$$\tilde{E} = \hat{x}\tilde{E}_x^+(z) + \hat{y}\tilde{E}_y^+(z)$$

$$\tilde{H} = \hat{x}\tilde{H}_x^+(z) + \hat{y}\tilde{H}_y^+(z)$$

$$\hat{k} = \hat{z}$$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \frac{1}{\eta}\hat{z} \times \tilde{E} = -\underset{\hat{z}\times\hat{y}}{\hat{x}}\frac{\tilde{E}_y^+(z)}{\eta} + \underset{\hat{z}\times\hat{x}}{\hat{y}}\frac{\tilde{E}_x^+(z)}{\eta}$$



$$\tilde{H}_x^+(z) = -\frac{\tilde{E}_y^+(z)}{\eta} \qquad \tilde{H}_y^+(z) = \frac{\tilde{E}_x^+(z)}{\eta}$$

Wave propagation example 1:

Magnetic field phasor of a plane wave in medium with

intrinsic impedance $\eta = 100\Omega$ is given by: $\widetilde{H} = (\hat{y}10 + \hat{z}20)e^{-j4x}$ $[mA/m]$

Find associated electric field phasor.

$$\widetilde{H} = (\hat{y}10 + \hat{z}20)e^{-j4x} \ [mA/m]$$

To find the electric field phasor we can use: $\tilde{E} = -\eta\hat{k} \times \widetilde{H}$

What is $\hat{k}$?

From $e^{-j4x} \rightarrow$ we know $\hat{k} = \hat{x}$ propagation direction

mA → A

$$\tilde{E} = -\eta\hat{k} \times \widetilde{H} = -100[\hat{x} \times (\hat{y}10 + \hat{z}20)]e^{-j4x} \times 10^{-3}$$

$$\tilde{E} = -[\hat{z} + \hat{y}2]e^{-j4x} \quad [V/m]$$

Wave propagation example 2:

Suppose now: $\widetilde{H} = \hat{y}\left(10e^{-j3x} - 20e^{j3x}\right)\ [mA/m]$

Find the associated electric field phasor:

Note magnetic field has 2 components
$$\begin{cases} e^{-j3x} \rightarrow \text{traveling in } + \hat{x} \\ e^{j3x} \rightarrow \text{traveling in } - \hat{x} \end{cases}$$

$$\widetilde{H} = \widetilde{H}_1 + \widetilde{H}_2 \qquad \widetilde{H}_1 = \hat{y}\left(10e^{-j3x}\right) \qquad \widetilde{H}_2 = -\hat{y}\left(20e^{j3x}\right)$$

$$\tilde{E}_1 = -\eta\hat{k} \times \widetilde{H}_1 = -100(\hat{x} \times \hat{y})10e^{-j3x} \times 10^{-3} = -\hat{z}e^{-j3x}\ [V/m]$$

$$\tilde{E}_2 = -\eta\hat{k} \times \widetilde{H}_2 = -100(-\hat{x}) \times (-\hat{y})20e^{j3x} \times 10^{-3} = -\hat{z}2e^{j3x}\ [V/m]$$

$$\tilde{E} = \tilde{E}_1 + \tilde{E}_2 = -\hat{z}\left(e^{-j3x} + 2e^{j3x}\right)\ [V/m]$$

Wave polarization

Polarization – describes <u>locus</u> traced by tip of $\vec{E}$ vector in space (within plane perpendicular to direction of propagation) as function of time

Most general → elliptical polarization

circular linear

z – components of E, H = 0 if $\hat{z}$ is propagation direction

$\tilde{E}(z)$ with $\hat{z}$ propagation: $\tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z)$

$\tilde{E}_x = E_{x0}e^{-jkz}$ $\tilde{E}_y = E_{y0}e^{-jkz}$ (Only propagate in +z direction)

$E_{x0}, E_{y0} \rightarrow$ in general composed with magnitude and phase

Phase is defined relative to reference, such as z=0, t=0.

Polarization depends on phase and amplitude of E_{x0} relative to E_{y0}

Wave polarization

We set phase of E_{x0} as zero for reference

δ = phase of E_{y0} <u>relative</u> to E_{x0} = phase difference

$$\begin{cases} E_{x0} = a_x & a_x = |E_{x0}| \geq 0 \\ E_{y0} = a_y e^{j\delta} & a_y = |E_{y0}| \geq 0 \end{cases}$$

$$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z) = (\hat{x}a_x + \hat{y}a_y e^{j\delta})e^{-jkz}$$

Instantaneous field: $\vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right]$

$$\vec{E}(z,t) = \hat{x}a_x \cos(\omega t - kz) + \hat{y}a_y \cos(\omega t - kz + \delta)$$

Magnitude of $\vec{E}(z,t)$:

$$\left|\vec{E}(z,t)\right| = \sqrt{E_x^2(z,t) + E_y^2(z,t)} = ({a_x}^2 cos^2(\omega t - kz) + {a_y}^2 cos^2(\omega t - kz + \delta))^{1/2}$$

The direction of $\vec{E}(z,t)$ at specific position, z is given by "inclination angle" ψ :

$$\psi(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right]$$